Representations of finite sets and correspondences

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joint work with

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Correspondences, Relations

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Let $\mathcal{D}$ be an essentially small k-linear category

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  $\forall \varphi \in End_{\mathcal{D}}(X)$, $\forall m \in F(x)$, $\varphi m := F(\varphi)(m)$.

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and for $\psi \in \mathcal{D}(Z,Y)$, $\varphi \in \mathcal{D}(Y,X)$, $v \in V$,
$$L_{X,V}(\psi)(\varphi \otimes v) = (\psi \circ \varphi) \otimes v.$$

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- Conversely, if S is a simple representation of $\mathcal{D}$ over k

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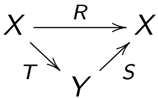

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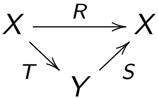

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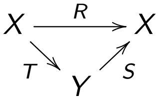

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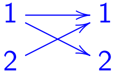
  If $R =$ 

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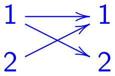
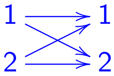
  If $R =$ 

  , then $R^2 =$ 

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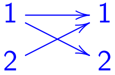
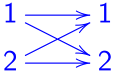
If $R =$ 

, then $R^2 =$ 

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Proof (sketch): Let $S \supseteq \Delta$, and $m \in \mathbb{N} - \{0\}$ such that $S^m = \overline{S}$.

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- If S is reflexive, then $\Delta \subseteq S \subseteq S^2 \subseteq \ldots \subseteq S^m = S^{m+1}$. This limit is the transitive closure of S, denoted by $\overline{S}$. It is a preorder.
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